European Journal of Combinatorics 33 (2012) 1965–1973

Contents lists available at SciVerse ScienceDirect

European Journal of Combinatorics

journal homepage: www.elsevier.com/locate/ejc

Countable homogeneous linearly ordered posets

Igor Dolinka, Dragan Mašulović

Department of Mathematics and Informatics, University of Novi Sad, Trg Dositeja Obradovića 4, 21101 Novi Sad, Serbia

ARTICLE INFO

Article history:
Received 2 November 2011
Received in revised form
14 June 2012
Accepted 18 June 2012
Available online 15 July 2012

ABSTRACT

A relational structure is called homogeneous if each isomorphism between its finite substructures extends to an automorphism of that structure. A linearly ordered poset is a relational structure consisting of a partial order relation on a set, along with a total (linear) order that extends the partial order in question. We characterise all countable homogeneous linearly ordered posets, thus extending earlier work by Cameron on countable homogeneous permutations. As a consequence of our main result it turns out that, up to isomorphism, there is a unique homogeneous linear extension of the random poset, the unique countable homogeneous universal partially ordered set.

1. Introduction

A first-order structure $\mathcal{A}$ is called *homogeneous* (sometimes also *ultrahomogeneous*) if any isomorphism between its finitely generated substructures extends to an automorphism of $\mathcal{A}$. In the case when we are dealing with relational structures (as we indeed do in this paper), the words 'finitely generated' may be simply replaced by 'finite'. As amply shown in [14], homogeneous (relational) structures constitute a very active and broad field of research, which, despite the model-theoretical framework of the definition of homogeneity, belongs essentially to combinatorics, with significant connections to permutation group theory, descriptive set theory, topology and, more recently, to semigroups, universal algebra, and theoretical computer science. One of the most basic and most important tasks set forth by this theory is to classify homogeneous members in various natural classes of structures. For instance, nowadays we have characterisations of homogeneous structures belonging to the following classes: finite graphs [7], countable partially ordered sets [15], countably infinite graphs [13], countable tournaments [12], countable digraphs [2], finite groups [3], countable permutations [1], and countable multipartite graphs [9]. The present paper is a contribution towards this classification programme. Taking [1,15] as our starting points, we provide here a description of all countable homogeneous *linearly ordered posets* (*lo-posets* for short): relational structures of the form

E-mail addresses: dockie@dmi.uns.ac.rs (I. Dolinka), masul@dmi.uns.ac.rs (D. Mašulović).

0195-6698/$ – see front matter © 2012 Elsevier Ltd. All rights reserved.
doi:10.1016/j.ejc.2012.06.006

$\mathcal{A} = (A, \prec, \sqsubset)$, where $\prec$ is a (strict) partial order of the set A, and $\sqsubset$ is a *linear extension* of $\prec$, a total order such that $\prec \subseteq \sqsubset$.

Totally ordered first-order structures (and, in particular, lo-posets) make a key appearance in the study of Ramsey properties of general structures, thus providing a link to important questions of topological dynamics, such as the calculation of the universal minimal flow of a topological group and the study of (extremely) amenable groups, which may or may not arise as full automorphism groups of countable homogeneous structures (i.e. Fraïssé limits [6,8]). The interested reader is directed e.g. to [10,11,16,17] for more details about this connection. For a general background, we refer to [4] for ordered sets, and to [8] for model theory.

The paper is organised as follows. In the following section, we introduce several definitions and some notation concerning ordered sets that we use throughout the paper; also, we define a number of small lo-posets that will play a central role in obtaining the required classification. Then, our discussion splits into two basic cases. The first of them, discussed in Section 3, concerns linearly ordered posets constructed by recasting the traditional view of a permutation as a structure equipped with two total orders into a "splitting" (that is, an edge colouring) of a single linear order on a set into two complementary posets on the same set. (This approach was already utilised with success in [5] in the course of obtaining a full description of finite homomorphism-homogeneous permutations.) As we have categorical equivalence here, it then suffices to invoke Cameron's characterisation of countable homogeneous permutations from [1] to complete the analysis in this case. The other, "non-permutational" case (equivalent to the presence of a particular three-element substructure called I^*) is considered in Section 4. So, apart from the homogeneous lo-posets arising from countable homogeneous permutations, we obtain here two additional families of countable homogeneous lo-posets: the first consists of $\kappa \rtimes \mathbb{Q}$, for $2 \leq \kappa \leq \aleph_0$, which should be thought of as a "homogeneous mixture" of κ copies of the usual order of rationals $(\mathbb{Q}, <)$ "shuffled" into a single linear order, while the second has a single member, the *random lo-poset* $\mathcal{L}$, the Fraïssé limit of all finite lo-posets. Since our exposition is mainly heuristic, our main result, Theorem 4.8, appears at the end of our argument, summarising the previous considerations. Finally, as an application, we prove in Section 5 that for the *random poset* $\mathcal{P}$, the unique countable homogeneous universal poset, there is, up to isomorphism, a unique linear extension such that the resulting lo-poset is homogeneous: this is precisely $\mathcal{L}$.

2. Basic notions and seven small lo-posets

Recall that a *strict* partial order on A is an irreflexive and transitive binary relation $\prec$. All the ordering relations in this paper are strict ordering relations.

As already defined in the introduction, a *linearly ordered poset* (that is, a *lo-poset*) is an ordered triple $\mathcal{A} = (A, \prec, \sqsubset)$ where $(A, \prec)$ is a poset and $(A, \sqsubset)$ is a linear extension of $(A, \prec)$, that is, $\sqsubset$ is a linear order on A such that $\prec \subseteq \sqsubset$.

We write $x \parallel y$ to denote that x and y are incomparable with respect to $\prec$.

Fig. 1 depicts seven small lo-posets. Solid lines constitute the Hasse diagram of the first ordering relation $\prec$, while a dotted arrow going from x to y indicates that $x \parallel y$ (with respect to $\prec$) and $x \sqsupset y$. These small structures will play a central role in distinguishing the cases in the course of proving our main result, Theorem 4.8.

Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a lo-poset and $\varnothing \neq B \subseteq A$. Then $\mathcal{A}[B] = (B, \prec \cap B^2, \sqsubset \cap B^2)$ is the *substructure of* $\mathcal{A}$ *induced by* B. For a lo-poset $\mathcal{A}'$ we write $\mathcal{A}' \hookrightarrow \mathcal{A}$ if $\mathcal{A}' \cong \mathcal{A}[B]$ for some non-empty $B \subseteq A$.

Let $(A, \prec_A)$ and $(B, \prec_B)$ be linearly ordered sets. The *lexicographic order* $\lhd$ on $A \times B$ (with respect to $\prec_A$ and $\prec_B$) is defined as follows: $(a, b) \lhd (a', b')$ if and only if either $a \prec_A a'$, or $a = a'$ and $b \prec_B b'$. By $\lhd_{\mathbb{Q}}$ we denote the lexicographic order on $\mathbb{Q} \times \mathbb{Q}$ with respect to the usual ordering of rationals.

3. Countable lo-posets arising from permutations

In this section, we show that homogeneous countable lo-posets that omit I^* are precisely the lo-posets that arise from homogeneous countable permutations described by Cameron in [1]. Recall that a permutation of A can be thought of as a relational structure $(A, \sqsubset_1, \sqsubset_2)$ where both $\sqsubset_1$ and $\sqsubset_2$ are linear orders on A (see [1,5]).

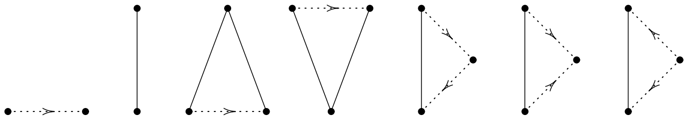

Fig. 1. Seven small lo-posets.

Each permutation $(A, \sqsubset_1, \sqsubset_2)$ gives rise to a lo-poset $(A, \prec, \sqsubset_2)$ where $\prec = \sqsubset_1 \cap \sqsubset_2$. Conversely, for a lo-poset $\mathcal{A} = (A, \prec, \sqsubset)$ define a binary relation $\ll$ as follows: $x \ll y$ if $x \prec y$, or $x \parallel y$ and $y \sqsubset x$. We say that $\mathcal{A}$ *arises from a permutation* if $\ll$ is transitive. It is easy to see that if $x \not\ll y$ and $x \neq y$ then $y \ll x$; consequently, if $\ll$ is transitive then it is a linear order on A.

Lemma 3.1. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a lo-poset. Then $I^* \not\hookrightarrow \mathcal{A}$ if and only if $\mathcal{A}$ arises from a permutation.*

Proof. ($\Leftarrow$) Assume that $I^* \hookrightarrow \mathcal{A}$ and let $a, b, c \in A$ be elements of $\mathcal{A}$ satisfying $a \prec c$, $a \sqsubset b \sqsubset c$, $b \parallel a$ and $b \parallel c$. Then $c \ll b$ and $b \ll a$, but $c \not\ll a$. Therefore, $\ll$ is not transitive.

($\Rightarrow$) Assume that $\ll$ is not transitive and take any $a, b, c \in A$ such that $a \ll b$ and $b \ll c$ but $a \not\ll c$. There are several easy cases to consider.

Assume first that $a \prec b$. Then $b \not\prec c$ (because $b \prec c$ implies $a \prec c$ and hence $a \ll c$), so $b \parallel c$ and $c \sqsubset b$. From $a \not\ll c$ it follows that $c \ll a$. It is not the case that $c \prec a$ (for then we would have $c \prec b$, which contradicts $c \parallel b$), so $a \parallel c$ and $a \sqsubset c$, thus showing $\mathcal{A}[a, b, c] \cong I^*$.

The reasoning is analogous in case $b \prec c$.

Assume, finally, that $a \not\prec b$ and $b \not\prec c$. Then $c \sqsubset b \sqsubset a$. From $a \not\ll c$ it follows that $c \ll a$. Therefore, $c \prec a$ and we again have $\mathcal{A}[a, b, c] \cong I^*$. □

Lemma 3.2. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a lo-poset such that $I^* \not\hookrightarrow \mathcal{A}$. Then $\mathcal{A}$ is a homogeneous lo-poset if and only if $\mathcal{A}' = (A, \ll, \sqsubset)$ is a homogeneous permutation.*

Proof. The proof is a straightforward consequence of the fact that the relations of each of the two structures are first-order definable in the other one in terms of quantifier-free formulae. For example, one possible definition of $\prec$ in $\mathcal{A}'$ is $x \prec y \Leftrightarrow x \ll y \wedge x \sqsubset y$, while $\ll$ may be defined in $\mathcal{A}$ by $x \ll y \Leftrightarrow x \prec y \vee (\neg(x \prec y) \wedge \neg(y \prec x) \wedge y \sqsubset x)$. □

Theorem 3.3. *Let $\mathcal{A}$ be a countable lo-poset such that $I^* \not\hookrightarrow \mathcal{A}$. Then $\mathcal{A}$ is homogeneous if and only if $\mathcal{A}$ arises from a homogeneous countable permutation.*

Consequently, $\mathcal{A}$ is isomorphic to one of the following five lo-posets:

1. $(\mathbb{Q}, <, <)$*, where $<$ is the usual ordering of $\mathbb{Q}$;*
2. $(\mathbb{Q}, =, <)$*, where $=$ is the equality relation on $\mathbb{Q}$;*
3. $(\mathbb{Q} \times \mathbb{Q}, \prec, \lhd_{\mathbb{Q}})$ *where $\prec$ is defined by $(a, b) \prec (c, d)$ if and only if $a = c$ and $b < d$ in $\mathbb{Q}$;*
4. $(\mathbb{Q} \times \mathbb{Q}, \prec, \lhd_{\mathbb{Q}})$ *where $\prec$ is defined by $(a, b) \prec (c, d)$ if and only if $a < c$ in $\mathbb{Q}$;*
5. $(\mathbb{Q} \times \mathbb{Q}, \prec, \sqsubset)$ *where $\prec$ and $\sqsubset$ are defined as follows, for $\alpha, \beta \in \mathbb{R} \setminus \mathbb{Q}$ satisfying $\alpha \neq \beta$ and $\alpha \neq -\beta$:*
   - *$(a, b) \prec (c, d)$ if and only if $a - c < \min\{\alpha(b - d), \beta(b - d)\}$ in $\mathbb{R}$,*
   - *$(a, b) \sqsubset (c, d)$ if and only if $a - c < \beta(b - d)$ in $\mathbb{R}$.*

Proof. Follows immediately from Lemmas 3.1 and 3.2, and the description of countable homogeneous permutations given in [1]. □

4. Countable non-permutational lo-posets

In this section, we assume that $\mathcal{A} = (A, \prec, \sqsubset)$ is a countable homogeneous lo-poset that embeds I^*. We start with a simple technical result.

Lemma 4.1. *Let $\mathcal{A}$ be a homogeneous lo-poset.*

(a) *If $\mathcal{A}$ embeds I^* then $\mathcal{A}$ embeds both I^+ and I^-.*
(b) *If $\mathcal{A}$ embeds I^* then: $\mathcal{A}$ embeds Λ if and only if $\mathcal{A}$ embeds V.*

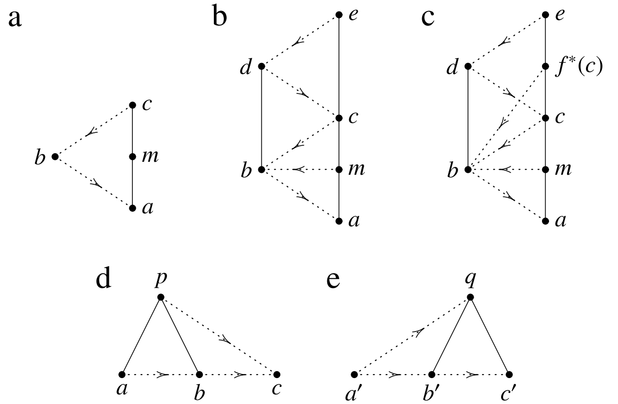

Fig. 2. The proof of Lemma 4.1.

Proof. (a) Consider the amalgam of a copy of I^* and a three-element chain as in Fig. 2(a). Clearly, $b \parallel m$. Assume that $b \sqsubset m$ (the other case follows by dual arguments). Then $\mathcal{A}[b, m, c] \cong I^+$. Let us show that $\mathcal{A}$ embeds I^- as well.

Consider the amalgam of this four-element structure and two copies of I^* as in Fig. 2(b). (Note that the relationships between b and e, m and d, and a and d in this amalgam are not specified at this point, and they are of no importance for the rest of the proof.) Then $f = \begin{pmatrix} b & e & m \\ b & e & c \end{pmatrix}$ is a local isomorphism, so it extends to an $f^* \in \mathrm{Aut}(\mathcal{A})$. Since $b \sqsubset c$, $b \parallel c$ and $m \prec c \prec e$, we have that $b \sqsubset f^*(c)$, $b \parallel f^*(c)$ and $c \prec f^*(c) \prec e$; see Fig. 2(c). Note, also, that $d \parallel f^*(c)$ because $d \parallel e$ and $d \parallel c$. Now, if $d \sqsubset f^*(c)$ then $\mathcal{A}[d, b, f^*(c)] \cong I^-$, while in case $d \sqsupset f^*(c)$ we have $\mathcal{A}[d, c, f^*(c)] \cong I^-$.

(b) It suffices to show direction ($\Rightarrow$) because direction ($\Leftarrow$) follows by dual arguments. Assume that $\mathcal{A}$ embeds Λ. By (a) we know that $\mathcal{A}$ embeds both I^+ and I^-. Take a copy of Λ and amalgamate it with a copy of I^+ to get the poset in Fig. 2(d). Similarly, take a copy of Λ and amalgamate it with a copy of I^- to get the poset in Fig. 2(e). If $a' \succ c'$ then $\mathcal{A}[a', c', q] \cong V$. If, however, $a' \parallel c'$ then $f = \begin{pmatrix} a & b & c \\ a' & b' & c' \end{pmatrix}$ is a local isomorphism (because $p \succ a$ and $p \parallel c$ implies $a \parallel c$), so it extends to an $f^* \in \mathrm{Aut}(\mathcal{A})$. Clearly, $a' \prec f^*(p)$, $b' \prec f^*(p)$, $c' \parallel f^*(p)$, $c' \sqsubset f^*(p)$ and $f^*(p) \parallel q$. Therefore, $\mathcal{A}[f^*(p), b', q] \cong V$. $\square$

4.1. Case 1

Assume first that $\mathcal{A}$ embeds neither Λ nor V. Then every connected component of $(A, \prec)$ is a chain, and $(A, \prec)$ is disconnected because $\mathcal{A}$ embeds I^*.

Lemma 4.2. *$(A, \sqsubset)$ is order-isomorphic to $\mathbb{Q}$.*

Proof. Since $\mathcal{A}$ embeds I^*, it follows that $\mathcal{A}$ embeds $\mathbf{2}$ (see Fig. 1), so the fact that $\mathcal{A}$ is homogeneous yields that $(A, \sqsubset)$ is without endpoints. Let us show that $(A, \sqsubset)$ is dense. By Lemma 4.1 (a) we know that $\mathcal{A}$ embeds I^+. Now, take any $x, z \in A$ such that $x \sqsubset z$. If $x \prec z$ then amalgamating I^* over x and z provides a y with $x \sqsubset y \sqsubset z$. If, however, $x \parallel z$ then amalgamating an appropriately chosen edge of I^+ over x and z provides a y with $x \sqsubset y \sqsubset z$. $\square$

Since $(A, \sqsubset)$ is order-isomorphic to $\mathbb{Q}$, it follows that $\mathcal{A}$ is a colouring of $\mathbb{Q}$, with connected components of $(A, \prec)$ being the maximal monochromatic subchains. This setting, reminiscent of that presented in [18], motivates the following definitions. We say that a countable lo-poset $\mathcal{A} = (A, \prec, \sqsubset)$ is a *coloured chain* if $(A, \prec)$ is disconnected and every connected component of $(A, \prec)$ is a chain. A coloured chain $\mathcal{A} = (A, \prec, \sqsubset)$ is *without endpoints* if for every $y \in A$ and every connected component

L of $(A, \prec)$ there exist $x, z \in L$ such that $x \sqsubset y \sqsubset z$; it is *dense* if for every $x, z \in A$ such that $x \sqsubset z$ and every connected component L of $(A, \prec)$ there exists a $y \in L$ such that $x \sqsubset y \sqsubset z$.

Lemma 4.3. *Let $\mathcal{A}_1 = (A_1, \prec_1, \sqsubset_1)$ and $\mathcal{A}_2 = (A_2, \prec_2, \sqsubset_2)$ be dense coloured chains without endpoints. Then*

(1) *$\mathcal{A}_1$ is homogeneous;*
(2) *$\mathcal{A}_1$ and $\mathcal{A}_2$ are isomorphic if and only if $(A_1, \prec_1)$ and $(A_2, \prec_2)$ have the same number of connected components.*

Proof. Both claims follow by a standard back-and-forth argument. $\square$

Let κ be a cardinal such that $2 \leqslant \kappa \leqslant \aleph_0$ and let $\mathbb{Q}_\kappa$ denote the κ-coloured dense chain constructed as in [18]: its domain is (order-isomorphic to) $\mathbb{Q}$ and is "interdensely coloured", that is, between any two points there are points of all possible colours. It exists and is unique up to isomorphism, because it is the Fraïssé limit of the class of all finite κ-coloured linear orders. Now, on the same base set $\mathbb{Q}$ define a lo-poset $\kappa \rtimes \mathbb{Q} = (\mathbb{Q}, \prec, \sqsubset)$ by taking for $\sqsubset$ just the linear ordering of $\mathbb{Q}$, and $x \prec y$ to mean that x and y have the same colour and $x \sqsubset y$.

Theorem 4.4. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a countable lo-poset embedding I^*, while omitting both Λ and V. Then $\mathcal{A}$ is homogeneous if and only if $\mathcal{A} \cong \kappa \rtimes \mathbb{Q}$ for some cardinal κ such that $2 \leqslant \kappa \leqslant \aleph_0$.*

Proof. ($\Leftarrow$) Clearly, $\kappa \rtimes \mathbb{Q}$ is a dense coloured chain without endpoints, so it is homogeneous by Lemma 4.3.

($\Rightarrow$) In the argument at the beginning of the section we have already concluded that $\mathcal{A}$ is a coloured chain. Let κ be the number of connected components of $(A, \prec)$. Note that $\kappa \geqslant 2$ since $\mathcal{A}$ embeds I^*. By Lemma 4.3 in order to prove $\mathcal{A} \cong \kappa \rtimes \mathbb{Q}$ it suffices to show that $\mathcal{A}$ is a dense coloured chain without endpoints.

The fact that $\mathcal{A}$ is without endpoints follows readily from the assumption that $\mathcal{A}$ embeds I^*. Let us show that $\mathcal{A}$ is dense. Take any $x, z \in A$ such that $x \sqsubset z$ and an arbitrary connected component L of $(A, \prec)$. We have to show that there exists a $y \in L$ such that $x \sqsubset y \sqsubset z$.

Let L_z be the connected component of $\mathcal{A}$ such that $z \in L_z$. Since $\mathcal{A}$ is without endpoints there is a $q \in L$ such that $q \sqsupset z$. Similarly, there are $r \in L_z$ and $s \in L$ such that $s \sqsupset r \sqsupset q$. Since $\mathcal{A}$ is homogeneous, the local automorphism $f = \begin{pmatrix} x & r & s \\ x & z & q \end{pmatrix}$ of $\mathcal{A}$ extends to some $f^* \in \mathrm{Aut}(\mathcal{A})$. Then $y = f^*(q)$ belongs to L and has the property that $x \sqsubset y \sqsubset z$. $\square$

4.2. Case 2

Assume now that $\mathcal{A}$ embeds Λ or V. Then by Lemma 4.1, $\mathcal{A}$ embeds both I^+ and I^- as well as both Λ and V.

Lemma 4.5. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a countable homogeneous lo-poset embedding Λ, V, I^-, I^+ and I^*. If $\mathcal{A}$ embeds a finite lo-poset $\mathcal{B} = (B, \prec, \sqsubset)$, then:*

(a) *$\mathcal{A}$ embeds the lo-poset $\mathcal{B}^1 = (B \cup \{1\}, \prec, \sqsubset)$ where $1 \notin B$ and $\forall b \in B\ (b \prec 1)$;*
(b) *$\mathcal{A}$ embeds the lo-poset $\mathcal{B}_0 = (B \cup \{0\}, \prec, \sqsubset)$ where $0 \notin B$ and $\forall b \in B\ (0 \prec b)$;*
(c) *$\mathcal{A}$ embeds the lo-poset $\mathcal{B}^- = (B \cup \{c\}, \prec, \sqsubset)$ where $c \notin B$ and $\forall b \in B\ (c \parallel b \wedge c \sqsupset b)$;*
(d) *$\mathcal{A}$ embeds the lo-poset $\mathcal{B}^+ = (B \cup \{c\}, \prec, \sqsubset)$ where $c \notin B$ and $\forall b \in B\ (c \parallel b \wedge c \sqsubset b)$.*

Proof. (a) Because $\mathcal{A}$ is homogeneous, it suffices to present a finite sequence of amalgamations that produces a finite poset embedding $\mathcal{B}$ and has a largest element with respect to $\prec$. Let $m_1, \ldots, m_n$ be the list of all maximal elements of $(B, \prec)$. First, let us amalgamate the bottom element of $\overline{\mathbf{2}}$ over m_1, and let us denote the top element of $\overline{\mathbf{2}}$ in this amalgam by t_1. If $n = 1$ we are done. Assume, therefore, that $n > 1$, that $m_1 \sqsupset m_2 \sqsupset \cdots \sqsupset m_n$, and that we have constructed t_k such that $t_k \succ t_j$ for all $1 \leqslant j \leqslant k$. If $t_k \succ m_{k+1}$ we can set $t_{k+1} = t_k$. If not, then $t_k \sqsupset m_{k+1}$ so by amalgamating the two bottom elements of Λ over t_k and m_{k+1} we obtain a new element t_{k+1} (the tip of Λ) satisfying $t_{k+1} \succ t_k$ and $t_{k+1} \succ m_{k+1}$.

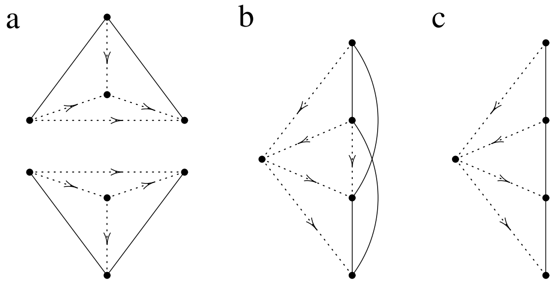

Fig. 3. Several small lo-posets.

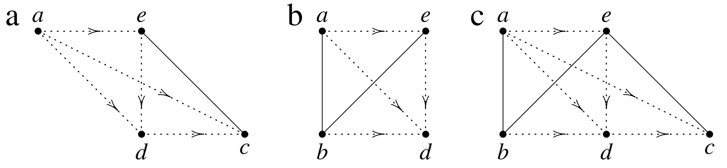

Fig. 4. The proof of Lemma 4.6(b).

(b) Dual to (a).
(c) Let $\mathcal{B}_0^1$ be the lo-poset obtained from $\mathcal{B}$ by adjoining the top element 1 and the bottom element 0. By (a) and (b) we have that $\mathcal{A}$ embeds $\mathcal{B}_0^1$. Also, note that $0 \prec 1$ in $\mathcal{B}_0^1$. Let us amalgamate the two comparable elements of I^- (with respect to $\prec$) over 0 and 1 in $\mathcal{B}_0^1$, and let us denote the third element of I^- in this amalgam by c. Then, clearly, $\forall b \in B$ ($c \parallel b \wedge c \sqsupset b$).
(d) Dual to (c). $\square$

Lemma 4.6. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a countable homogeneous lo-poset embedding Λ, V, I^-, I^+ and I^*.*

(a) *For every $n \geqslant 2$, $\mathcal{A}$ embeds the lo-poset $\mathcal{B}_n = (\{1, 2, \dots, n\}, \prec, \sqsubset)$ where $i \not\prec j$ for all $i, j \in \{1, 2, \dots, n\}$ and $1 \sqsubset 2 \sqsubset \cdots \sqsubset n$.*
(b) *$\mathcal{A}$ embeds the four lo-posets depicted in Fig. 3.*
(c) *Assume that $\mathcal{A}$ embeds a finite lo-poset $\mathcal{B} = (B, \prec, \sqsubset)$ where $B = \{b_1, b_2, \dots, b_n\}$ and $b_1 \sqsupset b_2 \sqsupset \cdots \sqsupset b_n$. Then for every $i \in \{2, \dots, n\}$, $\mathcal{A}$ embeds the lo-poset $\mathcal{B}^{(i)} = (B \cup \{c\}, \prec, \sqsubset)$ where $c \notin B$, $\forall b \in B$ ($c \parallel b$) and $b_1 \sqsupset \cdots \sqsupset b_{i-1} \sqsupset c \sqsupset b_i \sqsupset \cdots \sqsupset b_n$.*

Proof. (a) Clearly, $\mathcal{A}$ embeds $\mathcal{B}_2 \cong \mathbf{2}$. Assume now that $\mathcal{A}$ embeds $\mathcal{B}_k$ for some $k \geqslant 2$. Then, by Lemma 4.5, $\mathcal{A}$ also embeds $(\mathcal{B}_k)^+ \cong \mathcal{B}_{k+1}$.
(b) Let us first prove that $\mathcal{A}$ embeds the first of the two lo-posets in Fig. 3(a). The proof for the other one then follows by dual arguments.
By Lemma 4.5 we know that $\mathcal{A}$ embeds the four-element structure $(I^*)^-$ (Fig. 4(a)) as well as the four-element structure V^+ (Fig. 4(b)). By amalgamating these two structures over the common substructure induced by $\{a, d, e\}$ we obtain that $\mathcal{A}$ embeds the five-element structure depicted in Fig. 4(c). The only relationship in this amalgam that is not obvious is the one between b and c. However, $c \sqsubset b$ (since $c \sqsubset d$ and $d \sqsubset b$) and $c \parallel b$ (since $c \prec b$ would imply $c \prec a$, which is not the case). It is now easy to see that the substructure of this five-element structure induced by $\{b, c, d, e\}$ is isomorphic to the first of the two lo-posets in Fig. 3(a).
Then $\mathcal{A}$ also embeds the lo-poset in Fig. 3(b), which is easily seen to be an amalgam of the two lo-posets depicted in Fig. 3(a) (amalgamated over the common substructure consisting of three mutually incomparable elements with respect to $\prec$), as well as the lo-poset in Fig. 3(c), which can easily be obtained by amalgamating a copy of I^+ on top of a copy of I^*, on top of a copy of I^-.
(c) Let $\mathcal{B}_0^1$ be the lo-poset obtained from $\mathcal{B}$ by adjoining the top element 1 and the bottom element 0. By Lemma 4.5 we have that $\mathcal{A}$ embeds $\mathcal{B}_0^1$. If $b_{i-1} \succ b_i$ let us amalgamate the four-element chain

(with respect to $\prec$) of the lo-poset in Fig. 3(c) over $0 \prec b_i \prec b_{i-1} \prec 1$ and let us denote the fifth element of this amalgam by c. If, however, $b_{i-1} \parallel b_i$ let us amalgamate the four "vertical" elements of the lo-poset in Fig. 3(b) over $0 \prec b_i \sqsubset b_{i-1} \prec 1$ and let us denote the fifth element of this amalgam by c. In both cases we have $\forall b \in B\ (c \parallel b)$ and $b_1 \sqsupset \cdots \sqsupset b_{i-1} \sqsupset c \sqsupset b_i \sqsupset \cdots \sqsupset b_n$. $\square$

The proof of the following theorem bears a resemblance to the proof of the main result of [15]. Still, several technical aspects are different.

Theorem 4.7. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a countable lo-poset embedding I^* and at least one of Λ and V. Then $\mathcal{A}$ is homogeneous if and only if $\mathcal{A}$ is the Fraïssé limit of the class of all finite lo-posets.*

Proof. ($\Leftarrow$) Obvious.

($\Rightarrow$) As we have seen at the beginning of the subsection, if $\mathcal{A}$ embeds I^* and embeds Λ or V then by Lemma 4.1, $\mathcal{A}$ embeds I^+, I^-, Λ and V.

Let us show that $\mathcal{A}$ embeds every finite lo-poset. For a finite lo-poset $\mathcal{B} = (B, \prec, \sqsubset)$ let $m_{\mathcal{B}}$ denote the number of maximal elements in $(B, \prec)$, and let $\nu_{\mathcal{B}}$ denote the number of nonmaximal elements in $(B, \prec)$, so that $m_{\mathcal{B}} + \nu_{\mathcal{B}} = |B|$. We prove the theorem by induction on $\nu_{\mathcal{B}}$.

If $\nu_{\mathcal{B}} = 0$ then $(B, \prec)$ is an antichain, so $\mathcal{A}$ embeds $\mathcal{B}$ by Lemma 4.6(a).

Assume that the claim is true for all finite lo-posets $\mathcal{C}$ with $\nu_{\mathcal{C}} < k$ and let $\mathcal{B}$ be a finite lo-poset with k nonmaximal elements in $(B, \prec)$. The proof now proceeds by induction on $m_{\mathcal{B}}$.

Assume first that $m_{\mathcal{B}} = 1$. Let x be the only maximal element in $(B, \prec)$ and let $\mathcal{B}' = \mathcal{B}[B \setminus \{x\}]$ be the lo-poset obtained from $\mathcal{B}$ by removing x. Clearly, $\nu_{\mathcal{B}'} < \nu_{\mathcal{B}}$, so $\mathcal{A}$ embeds $\mathcal{B}'$ by the induction hypothesis (of the "outer" induction that runs on $\nu_{\mathcal{B}}$). Lemma 4.5 then ensures that $\mathcal{A}$ embeds $(\mathcal{B}')^1 \cong \mathcal{B}$.

Assume now that the claim is true for all finite lo-posets $\mathcal{C}$ with $\nu_{\mathcal{C}} = k$ and $m_{\mathcal{C}} < l$ for some l and let $\mathcal{B}$ be a finite lo-poset with k nonmaximal elements and l maximal elements in $(B, \prec)$.

If there is an element $y \in B$ that is both maximal and minimal in $(B, \prec)$, consider $\mathcal{B}' = \mathcal{B}[B \setminus \{y\}]$. Clearly, $m_{\mathcal{B}'} < l$, so $\mathcal{A}$ embeds $\mathcal{B}'$ by the induction hypothesis (of the "inner" induction that runs on $m_{\mathcal{B}}$). Then Lemma 4.6(c) ensures that $\mathcal{A}$ also embeds $\mathcal{B}$.

Next, suppose that no $x \in B$ is both a maximal and a minimal element of $(B, \prec)$. If $(B, \prec)$ has a unique minimal element $z \in B$, let $\mathcal{B}' = \mathcal{B}[B \setminus \{z\}]$ be the lo-poset obtained from $\mathcal{B}$ by removing z. Clearly, $\nu_{\mathcal{B}'} < \nu_{\mathcal{B}}$, so $\mathcal{A}$ embeds $\mathcal{B}'$ by the induction hypothesis. Lemma 4.5 then ensures that $\mathcal{A}$ embeds $(\mathcal{B}')_0 \cong \mathcal{B}$.

Finally, assume that there are at least two minimal elements in $(B, \prec)$ and let $a_1 \sqsupset \cdots \sqsupset a_s$, $s \geqslant 2$, be the list of all minimal elements in $(B, \prec)$ ordered by $\sqsupset$. Let p and q be two distinct elements not in B and define $\prec^*$ and $\sqsubset^*$ on $D = B \cup \{p, q\}$ as follows:

- $\prec^* = \prec \cup \{(a_1, p)\}$;
- $\sqsubset^* = \sqsubset \cup \{(x, p) : x \in B\} \cup \{(q, x) : x \in B \setminus \{a_s\}\} \cup \{(a_s, q)\}$.

Let $\mathcal{D} = (D, \prec^*, \sqsubset^*)$, $\mathcal{D}_1 = \mathcal{D}[D \setminus \{a_1\}]$ and $\mathcal{D}_2 = \mathcal{D}[D \setminus \{a_s\}]$. Since $\nu_{\mathcal{D}_1} < \nu_{\mathcal{B}}$ and $\nu_{\mathcal{D}_2} < \nu_{\mathcal{B}}$ the induction hypothesis ensures that $\mathcal{A}$ embeds both $\mathcal{D}_1$ and $\mathcal{D}_2$. It is now easy to see that $\mathcal{B}$ is a substructure of the amalgam of $\mathcal{D}_1$ and $\mathcal{D}_2$ (the amalgamation is over the common substructure $\mathcal{D}[D \setminus \{a_1, a_s\}]$; note also that the exact relationship between p and q in this amalgam is of no importance for the conclusion). Therefore, $\mathcal{A}$ embeds $\mathcal{B}$. $\square$

4.3. *Summary: the characterisation theorem*

Let us now summarise the results in this paper. By collecting the results of Theorems 3.3, 4.4 and 4.7 we obtain the following classification of countable homogeneous lo-posets.

Theorem 4.8. *Let $\mathcal{A} = (A, \prec, \sqsubset)$ be a countable lo-poset. Then $\mathcal{A}$ is homogeneous if and only if $\mathcal{A}$ is isomorphic to one of the following countable lo-posets:*

(1) *a lo-poset arising from a countable homogeneous permutation, as described in Theorem 3.3;*
(2) *$\kappa \rtimes \mathbb{Q}$ for some cardinal κ such that $2 \leqslant \kappa \leqslant \aleph_0$;*
(3) *the Fraïssé limit $\mathcal{L}$ of the class of all finite lo-posets.*

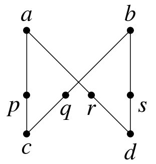

Fig. 5. The proof of Lemma 5.2.

5. Linear extensions of the random poset

As announced in the introduction, here we present an application of Theorem 4.8. We show that for the countable universal homogeneous poset $\mathcal{P}$, the Fraïssé limit of the class of all finite posets [8,15] (sometimes called the *random poset*), there is essentially a unique way to extend its order relation to a total order such that the resulting lo-poset is homogeneous: if $\mathcal{P} = (P, \prec)$ and if $\sqsubset_1$ and $\sqsubset_2$ are two linear extensions of $\prec$ such that both $(P, \prec, \sqsubset_1)$ and $(P, \prec, \sqsubset_2)$ are homogeneous, then $(P, \prec, \sqsubset_1) \cong (P, \prec, \sqsubset_2)$. This conclusion is reached in Corollary 5.3.

At this point, let us recall a few standard notions. A (lo-)poset $\mathcal{C}$ with the base set C is a *one-point extension* of a (lo-)poset $\mathcal{B}$ with the base set B if $C = B \cup \{x\}$ for some $x \notin B$. It is easy to see that a (lo-)poset $\mathcal{L}$ is homogeneous and universal for the class of all finite (lo-)posets, and thus the Fraïssé limit of the class of all finite (lo-)posets, if and only if $\mathcal{L}$ *realises all one-point extensions*, that is, for all finite (lo-)posets $\mathcal{B}$, $\mathcal{C}$ such that $\mathcal{C}$ is a one-point extension of $\mathcal{B}$, and for every embedding $f : \mathcal{B} \hookrightarrow \mathcal{L}$ there is an embedding $g : \mathcal{C} \hookrightarrow \mathcal{L}$ such that $g|_B = f$. (See [8] for details.)

Lemma 5.1. *Let $\mathcal{L} = (L, \prec, \sqsubset)$ be the Fraïssé limit of the class of all finite lo-posets (case* (3) *in Theorem 4.8). Then $\mathcal{L}' = (L, \prec)$ is the random poset, that is, the Fraïssé limit of the class of all finite posets.*

Proof. Let us show that $\mathcal{L}'$ realises all one-point extensions. Let $\mathcal{B}' = (B, \prec_B)$ be an arbitrary finite poset, let $\mathcal{C}' = (C, \prec_C)$ be a one-point extension of $\mathcal{B}'$ where $C = B \cup \{c\}$, and let $f : \mathcal{B}' \hookrightarrow \mathcal{L}'$ be an arbitrary embedding. Define $\sqsubset_B$ on B as follows: $b \sqsubset_B b'$ if and only if $f(b) \sqsubset f(b')$ in $\mathcal{L}$. Then, clearly, f is an embedding of the lo-poset $\mathcal{B} = (B, \prec_B, \sqsubset_B)$ into $\mathcal{L}$.

Assume that the following relations hold in $\mathcal{C}'$: $b_{j_1}, \dots, b_{j_l} \prec_C c \prec_C b_{i_1}, \dots, b_{i_k}$, with $c \parallel b$ in $\mathcal{C}'$ for all $b \in B \setminus \{b_{i_1}, \dots, b_{i_k}, b_{j_1}, \dots, b_{j_l}\}$. Then we have that $b_{j_t} \sqsubset_B b_{i_s}$ holds in $\mathcal{B}$ for all $1 \leqslant s \leqslant k$ and $1 \leqslant t \leqslant l$. Therefore, $\sqsubset_C$ defined by

$$\begin{aligned}\sqsubset_C \,=\, &\sqsubset_B \cup \{(c, x) : x \in B \text{ and } \exists s(b_{i_s} \sqsubseteq_B x)\}\\ &\cup\{(x, c) : x \in B \text{ and } \neg\exists s(b_{i_s} \sqsubseteq_B x)\}\end{aligned}$$

is a linear extension of $\prec_C$ such that the lo-poset $\mathcal{C} = (C, \prec_C, \sqsubset_C)$ is a one-point extension of $\mathcal{B}$. Since $\mathcal{L}$ is the Fraïssé limit of the class of all finite lo-posets, it realises all one-point extensions, so there is an embedding $g : \mathcal{C} \hookrightarrow \mathcal{L}$ such that $g|_B = f$. It is easy to see that g is then the embedding of the corresponding $\prec$-reducts, $g : \mathcal{C}' \hookrightarrow \mathcal{L}'$. This completes the proof that $\mathcal{L}'$ realises all one-point extensions. Therefore, $\mathcal{L}'$ is the Fraïssé limit of the class of all finite posets. □

Lemma 5.2. *Let $\mathcal{P} = (P, \prec)$ be the random poset and let $\sqsubset$ be an arbitrary linear extension of $\prec$. Then $I^* \hookrightarrow (P, \prec, \sqsubset)$.*

Proof. Let $\mathcal{A} = (A, \prec)$ be the poset depicted in Fig. 5. Clearly, $\mathcal{P}$ embeds $\mathcal{A}$ since $\mathcal{P}$ is universal for all finite posets. Fix an embedding of $\mathcal{A}$ in $\mathcal{P}$ and in that particular embedding consider the relationships of (a, b) and (c, d) with respect to $\sqsubset$. If $b \sqsubset a$ and $c \sqsubset d$ then $c \sqsubset s \sqsubset a$, so $\mathcal{P}[a, s, c] \cong I^*$. The remaining three cases are dealt with by analogous arguments. □

Corollary 5.3. *Let $\mathcal{P} = (P, \prec)$ be the random poset. There is a unique (in the sense indicated at the beginning of the section) linear extension $\sqsubset$ of $\prec$ such that $(P, \prec, \sqsubset)$ is a countable homogeneous lo-poset.*

Proof. Let $\mathcal{P} = (P, \prec)$ be the random poset and let $\sqsubset$ be a linear extension of $\prec$ such that $\mathcal{P}' = (P, \prec, \sqsubset)$ is a countable homogeneous lo-poset. Lemma 5.2 shows that $\mathcal{P}'$ does not arise from a homogeneous permutation, while the class (2) from Theorem 4.8 is excluded by the fact that every connected component of the $\prec$-reduct of $\kappa \rtimes \mathbb{Q}$ is a chain. On the other hand, Lemma 5.1 ensures that (3) is a possibility. Uniqueness follows from the fact that the Fraïssé limit of the class of all lo-posets is unique up to isomorphism; therefore, $\mathcal{P}' \cong \mathcal{L}$. $\square$

Acknowledgments

Both authors gratefully acknowledge the support of the Ministry of Science and Technological Development of the Republic of Serbia through Grant No.174019. The research of the first author is also supported by a grant (Contract 114–451–2002/2011) of the Secretariat of Science and Technological Development of the Autonomous Province of Vojvodina.

The authors would also like to thank the two anonymous referees whose remarks have significantly improved the presentation of the results in this paper.

References

[1] P.J. Cameron, Homogeneous permutations, Electron. J. Combin. 9 (2) (2002) #R2. 9pp.
[2] G. Cherlin, The classification of countable homogeneous directed graphs and countable homogeneous n-tournaments, Mem. Amer. Math. Soc. 131 (621) (1998) xiv+161pp.
[3] G. Cherlin, U. Felgner, Homogeneous finite groups, J. Lond. Math. Soc. (2) 61 (2000) 784–794.
[4] B.A. Davey, H.A. Priestley, Introduction to Lattices and Order, second ed., Cambridge University Press, Cambridge, 2002.
[5] I. Dolinka, É. Jungábel, Finite homomorphism-homogeneous permutations via edge colourings of chains, Manuscript, p. 12 (submitted for publication).
[6] R. Fraïssé, Sur certains relations qui généralisent l'ordre des nombres rationnels, C. R. Acad. Sci. Paris 237 (1953) 540–542.
[7] A.D. Gardiner, Homogeneous graphs, J. Combin. Theory Ser. B 20 (1976) 94–102.
[8] W. Hodges, A Shorter Model Theory, Cambridge University Press, Cambridge, 1997.
[9] T. Jenkinson, J.K. Truss, D. Seidel, Countable homogeneous multipartite graphs, European J. Combin. 33 (2012) 82–109.
[10] A.S. Kechris, V.G. Pestov, S. Todorčević, Fraïssé limits, Ramsey theory and topological dynamics of automorphism groups, Geom. Funct. Anal. 15 (2005) 106–189.
[11] A.S. Kechris, M. Sokić, Dynamical properties of the automorphism groups of the random poset and random distributive lattice, Manuscript, p. 31.
[12] A.H. Lachlan, Countable homogeneous tournaments, Trans. Amer. Math. Soc. 284 (1984) 431–461.
[13] A.H. Lachlan, R.E. Woodrow, Countable ultrahomogeneous undirected graphs, Trans. Amer. Math. Soc. 262 (1980) 51–94.
[14] D. Macpherson, A survey of homogeneous structures, Discrete Math. 311 (2011) 1599–1634.
[15] J.H. Schmerl, Countable homogeneous partially ordered sets, Algebra Universalis 9 (1979) 317–321.
[16] M. Sokić, Ramsey properties of finite posets, Order 29 (2012) 1–30.
[17] M. Sokić, Ramsey properties of finite posets II, Order 29 (2012) 31–47.
[18] S. Torrezao de Sousa, J.K. Truss, Countable homogeneous coloured partial orders, Dissertationes Math. 455 (2008) 48.